Log-linearization

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1 Linearization

This note looks at linearization and log-linearization. If you want to understand the basics, read the whole document. If you are well trained in math, you can jump to the "Getting clever" section towards the end to get the basic idea.

1.1 Ordinary linearization

Taylors theorem states that a scalar function $f(x)$ can be approximated by

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x - \bar{x}).$$

Similarly, a function of n variables can be approximated to a first order by

$$f(x_1, ..., x_n) \approx f(\bar{x}_1, ..., \bar{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{x}_1, ..., \bar{x}_n)(x_i - \bar{x}_i).$$

Example 1 *Let* $f(x_1, x_2) = x_1 x_2 + x_1^3$ *such that*

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= x_2 + 3x_1^2 \\ \frac{\partial f}{\partial x_1} &= x_1 \end{aligned}$$

$$f(x_1, x_2) \approx f(\bar{x}_1, \bar{x}_2) + \left(\bar{x}_2 + 3\bar{x}_1^2\right)(x_1 - \bar{x}_1) + \bar{x}_1(x_2 - \bar{x}_2)$$

Let's now linearize (notice, NOT log-linearize) an Euler equation. Remember the program, we find the steady-state, and linearize the model around that steady-state.

Example 2

$$\frac{1}{I_t} = \beta E_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right)$$

We use brute force and linearize both sides of this equation. First, the steady state is

$$\frac{1}{\bar{I}} = \beta \left(\frac{\bar{C}}{\bar{C}} \right)^{-\sigma} \frac{1}{\bar{\Pi}}$$

Let us assume that there is a monetary policy rule in place with a gross inflation target of 1 which will imply $\bar{\Pi} = 1$*, which means that steady state inflation is zero and hence we see that* $\bar{I} = \beta^{-1}$*, which by the way will also be the steady state real interest rate. Now we linearize the left hand side. Think of the left hand side as* $f(I) = 1/I$*, with* $f'(I) = -\frac{1}{I^2}$*. Hence, Taylors theorem then states that*

$$\frac{1}{I_t} \approx \frac{1}{\bar{I}} - \frac{1}{\bar{I}^2} \left(I - \bar{I} \right)$$

Now we move to the right-hand side and let $g(C, C_1, \Pi) = \beta \left(\frac{C_1}{C} \right)^{-\sigma} \frac{1}{\Pi}$*. Hence,*

$$\begin{aligned} \frac{\partial g}{\partial C} &= \beta \sigma C_1^{-\sigma} C^{\sigma - 1} \Pi^{-1} \\ \frac{\partial g}{\partial C_1} &= -\beta \sigma C_1^{-\sigma - 1} C^{\sigma} \Pi^{-1} \\ \frac{\partial g}{\partial \Pi} &= -\beta \left(\frac{C_1}{C} \right)^{-\sigma} \Pi^{-2} \end{aligned}$$

Using that $\Pi = 1$ *and that* $\bar{C} = \bar{C}_1$ *we find*

$$\begin{aligned} g\left(\bar{C}, \bar{C}_1, \bar{\Pi}\right) &= \beta \\ \frac{\partial g}{\partial C}\left(\bar{C}, \bar{C}_1, \bar{\Pi}\right) &= \beta \sigma \bar{C}^{-1} \\ \frac{\partial g}{\partial C_1}\left(\bar{C}, \bar{C}_1, \bar{\Pi}\right) &= -\beta \sigma \bar{C}^{-1} \\ \frac{\partial g}{\partial \Pi}\left(\bar{C}, \bar{C}_1, \bar{\Pi}\right) &= -\beta \end{aligned}$$

Hence Taylors theorem gives

$$g(C, C_1, \Pi) \approx \beta + \beta \sigma \bar{C}^{-1} \left(C - \bar{C} \right) - \beta \sigma \bar{C}^{-1} \left(C_1 - \bar{C} \right) - \beta (\Pi - 1)$$

Combining these results, we find that the linearized Euler equation can be written

$$\frac{1}{\bar{I}} - \frac{1}{\bar{I}^2} \left(I_t - \bar{I} \right) = \beta + E_t \left(\beta \sigma \bar{C}^{-1} \left(C_t - \bar{C} \right) - \beta \sigma \bar{C}^{-1} \left(C_{t+1} - \bar{C} \right) - \beta \left(\Pi_{t+1} - 1 \right) \right)$$

Next, we tidy up using steady state relations. First, since $\frac{1}{\bar{I}} = \beta$ *we see that the constant on both sides drops out. This is a general result, if you don't get the* *constants to cancel, you have made an error somewhere. Furthermore, we can replace* $\frac{1}{\bar{I}^2}$ *with* β^2 *and hence we notice that we can divide through with* β *to find*

$$-\beta\left(I_t-\bar{I}\right)=E_t\left(\sigma\bar{C}^{-1}\left(C_t-\bar{C}\right)-\sigma\bar{C}^{-1}\left(C_{t+1}-\bar{C}\right)-\left(\Pi_{t+1}-1\right)\right)$$

This is something which resembles a "usual" Euler equation. But notice that we look at absolute differences between for example consumption and the steady state value. Hence, levels matters for interpretations of impulse-response functions. The typical way to proceed would be to define gaps and solve the full model, and compute impulse-response functions to the structural economic shocks. That is, for example, how does consumption react to a technology shock. But to make those results comprehencible, we would have to scale the result by the steady state. An alternative is to use the following trick:

$$\left(I_t-\bar{I}\right)=\bar{I}\left(\frac{I_t}{\bar{I}}-1\right)$$

The gap in the last parenthesis is now the percentage deviation from steady state. If we use the same trick with all variables we can rewrite the equation as

$$-\beta\bar{I}\left(\frac{I_t}{\bar{I}}-1\right)=E_t\left(\sigma\bar{C}^{-1}\bar{C}\left(\frac{C_t}{\bar{C}}-1\right)-\sigma\bar{C}^{-1}\bar{C}\left(\frac{C_{t+1}}{\bar{C}}-1\right)-\left(\Pi_{t+1}-1\right)\right)$$

which simplifies to (using $\beta\bar{I}=1$*)*

$$-\hat{I}_t=E_t\left(\sigma\hat{C}_t-\sigma\hat{C}_{t+1}-\hat{\Pi}_{t+1}\right)$$

$$\begin{aligned}\hat{I}_t &= \frac{I_t}{\bar{I}}-1\\ \hat{C}_t &= \frac{C_t}{\bar{C}}-1\end{aligned}$$

which can be rewritten as

$$\hat{C}_t=E_t\hat{C}_{t+1}-\sigma^{-1}\left(\hat{I}_t-\hat{\Pi}_{t+1}\right)$$

which is very similar to the Euler equation (10) on page 21 in Gali, if we are willing to agree that the percentage difference in our definition above can approximately be written as $\log I_t-\log\bar{I}$. That is, $\hat{I}_t=\log I_t-\log\bar{I}$. Formally, this can be done by noting that

$$\frac{I_t}{\bar{I}}=\exp\left(\log I_t-\log\bar{I}\right)$$

and since Taylors theorem states that $\exp(x)$ is approximately $1+x$ around 0 (which is the relevant point of approximation since we approximate the above around $\bar{I}$. We hence see that

$$\exp\left(\log I_t-\log\bar{I}\right)\approx 1+\log I_t-\log\bar{I}$$

and hence that

$$\hat{I}_t = \frac{I_t}{\bar{I}} - 1 \approx \log I_t - \log \bar{I}.$$

Using this rewrite, we see that we can replace the deviations from steady state to get

$$\log C_t - \log \bar{C} = E_t \left(\log C_{t+1} - \log \bar{C}\right) - \sigma^{-1} \left(\log I_t - \log \bar{I} - \log \Pi_{t+1}\right)$$

which simplifies to

$$\log C_t = E_t \log C_{t+1} - \sigma^{-1} \left(\log I_t - \log \bar{I} - \log \Pi_{t+1}\right)$$

This is exactly Galis equation (10).

A more direct way to arrive at this equation, without the step where we approximate the percentage change, is to instead rewrite the original model in terms of logs and then linearize. This is called log-linearization.

1.2 Log-linearization

We again start from the Euler equation

$$\frac{1}{I_t} = \beta E_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right).$$

Notice that $I_t = \exp(\log I_t)$, where we let $\log x$ mean the natural logarith of x, that is, with the base e. This is the convention used in Matlab. Use this to rewrite the above model as

$$\frac{1}{\exp(\log I_t)} = \beta E_t ((\exp(\log C_{t+1}) / \exp(\log C_t))^{-\sigma} \frac{1}{\exp(\log \Pi_{t+1})}$$

Using standard rules for logs (such as $\log x^\alpha = \alpha \log x$ etc.),

$$\exp(-\log I_t) = \beta E_t \left(\exp(-\sigma \log C_{t+1} + \sigma \log C_t - \log \Pi_{t+1})\right)$$

Define new variables according to $x = \log X$ to write this as

$$\exp(-i_t) = \beta E_t \exp(-\sigma c_{t+1} + \sigma c_t - \pi_{t+1})$$

Since we have only trivially manipulated this equation, the steady state remains the same, meaning $\bar{I} = \beta^{-1}$ which in logs means $\bar{\imath} = -\log \beta$.

Now we approximate the above equation, but directly in terms of for example i_t in deviation from $\bar{\imath}$. Brute force we do this variable by variable by applying Taylors theorem (this time I ignore the constant as you know this will drop out) to find

$$\begin{aligned} -\exp(-\bar{\imath})(i_t - \bar{\imath}) &= -\beta\sigma E_t \exp(-\sigma\bar{c} + \sigma\bar{c} - \bar{\pi})(c_{t+1} - \bar{c}) + \beta\sigma \exp(-\sigma\bar{c} + \sigma\bar{c} - \bar{\pi})(c_t - \bar{c}) \\ &\quad -\beta E_t \exp(-\sigma\bar{c} + \sigma\bar{c} - \bar{\pi})(\pi_{t+1} - \bar{\pi}) \end{aligned}$$

(where the equation streaches out over two lines!). Using that $\bar{\imath} = -\log\beta$, and that $\bar{\pi} = \log\bar{\Pi} = 0$ in the zero steady state inflation case, we can rewrite the above equation as

$$-\beta\left(i_t - \bar{\imath}\right) = -\beta\sigma E_t\left(c_{t+1} - \bar{c}\right) + \beta\sigma\left(c_t - \bar{c}\right) + \beta E_t\left(\pi_{t+1} - \bar{\pi}\right) \tag{1}$$

This time, we *define* the percentage deviation from steady state as

$$\hat{x}_t \equiv \log X_t - \log\bar{X}.$$

With these definitions, you can easily manipulate (1) to recover equation (10) in Gali.

2 Getting clever

Brute force always works, but is often tedious. Many times, there are clever tricks to use. Here follows a compressed introduction to log-linearization and some useful tricks.

We start by noting that

$$X = \exp\left(\log X\right)$$

hence,

$$\frac{\partial X}{\partial\log X} = \exp\left(\log X\right) = X$$

The chain-rule then implies that

$$\frac{\partial f\left(X\right)}{\partial\log X} = f'\left(X\right)X \tag{2}$$

and hence Taylors theorem gives

$$f\left(X\right) - f\left(\bar{X}\right) \approx f'\left(\bar{X}\right)\bar{X}\left(\log X - \log\bar{X}\right) = f'\left(\bar{X}\right)\bar{X}\hat{x}_t \tag{3}$$

Notice that if we do a total differential of $f\left(X\right)$, we get

$$\begin{aligned} & f'\left(\bar{X}\right)dX \\ & f'\left(\bar{X}\right)\bar{X}\frac{dX}{\bar{X}} \end{aligned} \tag{4}$$

Comparing (3) and (4) we arrive at the following very useful lemma.

Lemma 3 *Practical log-linearization. Assume an expression of the form* $g\left(X\right) = f\left(X\right)$ *where* X *is a vector. If we take a total differential, where the partials are evalutated at steady-state, and replace* $\frac{dX_i}{\bar{X}_i}$ *with* $\hat{x}_i \equiv \log X_i - \log\bar{X}$ *then this will be identical to the log-linearized equation.*

Example 4 *Simple real Euler-equation again. Assume that* $\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} R_t$. *Apply the above lemma to log-linearize this equation:*

$$
\begin{aligned}
-\frac{1}{\bar{C}^2} dC_t &= -\frac{\beta \bar{R}}{\bar{C}^2} E_t dC_{t+1} + \beta \frac{1}{\bar{C}} dR_t \\
-\frac{1}{\bar{C}} \frac{dC_t}{\bar{C}} &= -\frac{\beta \bar{R}}{\bar{C}} E_t \frac{dC_{t+1}}{\bar{C}} + \beta \frac{\bar{R}}{\bar{C}} \frac{dR_t}{\bar{R}} \\
-\hat{c}_t &= -E_t \hat{c}_{t+1} + \hat{r}_t
\end{aligned}
$$

Next, notice that

$$
\frac{\partial \log f(X)}{\partial X} = \frac{1}{f(X)} f'(X) \tag{5}
$$

Combining these results shows that

$$
\frac{\partial \log f(X)}{\partial \log X} = \frac{\partial \log f(X)}{\partial X} \frac{\partial X}{\partial \log X} = \frac{1}{f(X)} f'(X) X \tag{6}
$$

Suppose you have an equation of the form

$$
g(X) = f(Y)
$$

then you know that in steady state,

$$
g(\bar{X}) = f(\bar{Y}). \tag{7}
$$

Use (2) and Taylors theorem to log-linearize both sides to find

$$
f'(\bar{X}) \bar{X} \hat{x}_t = g'(\bar{Y}) \bar{Y} \hat{y}_t
$$

Then divide this equation with (7) to get

$$
\frac{f'(\bar{X}) \bar{X}}{f(\bar{X})} \hat{x}_t = \frac{g'(\bar{Y}) \bar{Y}}{g(\bar{Y})} \hat{y}_t
$$

Notice that you can use this for direct computation: calucate the partials, divide by the steady state of f, multiply with the steady state of $\bar{X}$ and there you have the coefficient in front of $\hat{x}_t$ in the log-linearization, without having to rewrite the model in the cumbersome $\exp(\log(X))$ notation.

Even better, (6) implies that you can find $\frac{1}{f(X)} f'(X)$ imediately by instead calculating $\frac{\partial \log f(X)}{\partial \log X}$. If $f(x)$ is log-separable in the arguments (as is often the case), this greatly reduces computational burden.

Example 5 *The Euler equation clever:*

$$
\frac{1}{I_t} = \beta E_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right).
$$

Notice that this equation is log-separable (ignoring the expectations term, which we come back to), and hence we can think of this as

$$f(I) = g(C, C_1, \Pi)$$

To be clever, we use the above approach and first take logs to find that

$$\log(LHS) = -\log I_t.$$

Then, we immidiately see that the coefficient in front of $\hat{\imath}_t$ *is* $\frac{\partial}{\partial \log I_t}(-\log I_t) = -1$. *Similarly, we log the right-hand side to find*

$$\log(RHS) = \log\beta - \sigma(\log C_{t+1} - \log C_t) - \log\Pi_{t+1}$$

and easily recover the coefficient in front of C_{t+1} *as* $-\sigma$. *Proceeding like this, we have showed that*

$$-\hat{\imath}_t = -\sigma E_t\hat{c}_{t+1} - \sigma\hat{c}_{t+1} - E_t\hat{\pi}_{t+1}$$

which again can be manipulated to recover equation (10) in Gali, but this time with very little effort.

For a more detailed exposition, see the following chapter by Harald Uhlig: http://www.sfu.ca/~kkasa/uhlig1.pdf