Solving rational expectations models

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1 The setup

We assume that we have a model that can be written in the following form:

$$\begin{bmatrix} x_{1,t+1} \\ E_t x_{2,t+1} \end{bmatrix} = A \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \tag{1}$$

where $x_{1,t}$ is a $n_1 \times 1$ vector with predetermined - or state - variables and $x_{2,t}$ is a $n_2 \times 1$ vector of jump variables.

Let us pause for a moment and think about what this representation means. For the state variables,

$$\begin{aligned} x_{1,t+1} &= A_{11}x_{1,t} + A_{12}x_{2,t} + \varepsilon_{t+1} \\ E_t x_{t+2} &= A_{21}x_{1,t} + A_{22}x_{2,t} \end{aligned}$$

The first equation illustrates what it means that x_1 is predetermined: once the values of the current states and jump variables are determined, $x_{1,t+1}$ is predetermined, up to the effect of the exogenous innovation. Perhaps the most natural example of a state variable is the capital stock. The capital accumulation equation is written

$$K_{t+1} = (1-\delta) K_t + I_t$$

where I_t is investment. In this model, I_t is the jump variable to be determined at time t. Once I_t is known, K_{t+1} is fully determined. This equation then fits in the notation of the system with $x_{1,t} = K_t$, $x_{2,t} = I_t$, $A_{11} = 1-\delta$, $A_{12} = 1$, $\varepsilon_{t+1} = 0$. Of course, further equations would be neded to close this model fully.

Example 1 *A reduced form of the NK model. Consider the NK Phillips curve, but for simplicity assume that the output-gap follows an exogenous process:*

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa x_t \\ x_t &= \rho x_{t-1} + \varepsilon_t \end{aligned}$$

We rewrite this model to fit the notation above:

$$\begin{bmatrix} x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\beta^{-1}\kappa & \beta^{-1} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}.$$

2 Definition of solution

What do we mean by a solution to the system of difference equations (1)? There are two, very related, definitions of equilibrium.

Definition 2 Solution. *A solution to (1), given an initial condition $x_{1,0}$ and a sequence of the vector of disturbances $\{\varepsilon_t\}_1^\infty$ are sequences of $\{x_{1,t}\}_1^\infty$ and $\{x_{2,t}\}_1^\infty$. In normal language, this just say that if we get initial conditions and a string of exogenous shocks, the solution will be how all the variables in the system evolve over time.*

Definition 3 Solution, alternative. *A solution to (1) is provided by the functions $f(x_{1,t})$ and $g(x_{1,t})$ such that*

$$\begin{aligned} x_{1,t+1} &= f(x_{1,t}) + \varepsilon_{t+1} \\ x_{2,t} &= g(x_{1,t}). \end{aligned}$$

Here we see that the functions summarize the solution concept in the first definition: given initial conditions and $\{\varepsilon_t\}_1^\infty$, f and g allows us to fully recover $\{x_{1,t}\}_1^\infty$ and $\{x_{2,t}\}_1^\infty$. f tells us how the state variables evolve as a function only of their own lag + shocks. g tells us how to recover the jump variables from the current states.

Example 4 *The model is simple enough to allow an analytical solution. We successively replace future inflation with a leaded Phillips-curve to recover*

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa x_t \\ \pi_t &= \beta E_t (\beta E_{t+1} \pi_{t+2} + \kappa x_{t+1}) + \kappa x_t \\ \pi_t &= \kappa x_t + \kappa \beta E_t x_{t+1} + \kappa \beta^2 E_t x_{t+2} + ... \\ \pi_t &= \kappa E_t \Sigma_{T=t}^{\infty} \beta^{T-t} x_T \end{aligned}$$

where we also impose the transversality condition $\lim_{T\to\infty} E_t \pi_T = 0$. Since $E_t x_T = \rho^{T-t} x_t$ we finally find

$$\begin{aligned} \pi_t &= \kappa E_t (\beta\rho)^{T-t} x_t \\ \pi_t &= \frac{\kappa}{1-\beta\rho} x_t \end{aligned}$$

In this example, hence, $f(x_t) = \rho x_t$, $g(x_t) = \frac{\kappa}{1-\beta\rho} x_t$. If we want to recover the sequences we start from an initial condition say $x_0 = 1$. We then find $x_1 = \rho * 1 + \varepsilon_1$, $x_2 = \rho x_1 + \varepsilon_2$, etc. and $\pi_1 = \frac{\kappa}{1-\beta\rho} x_1$ etc.

2.1 Solution method: Matrix decomposition

This section follows lecture notes by Paul Söderlind closely, who in turn builds on Klein (199?).

We first eliminate the shocks by taking conditional expectations of (1) to get

$$E_t \begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = A \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

Let us do a Schur decomposition of the matrix A.

Lemma 5 *Schur decomposition. Given matrix A there exist (possibly complex) matrices Z and T, where Z is unitary (such that $Z^H Z = I$) and T is upper triangular with the eigenvalues of A along the diagonal.*

$$A = ZTZ^H$$

where Z^H is the complex conjugate of Z. Replace A with the Schur form, where the matrices are reordered such that the eigenvalues appear on the diagonal of T in accending order (top left has eigenvalue smallest in absolute value).

$$E_t \begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = ZTZ^H \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} \tag{2}$$

If the number of eigenvalues with $abs(\lambda_i) < 1$ is exactly equal to n_1, then define

$$\begin{bmatrix} s_t \\ e_t \end{bmatrix} = Z^H \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} \tag{3}$$

premultiply (2) with Z^H, use that $Z^H Z = I$ and rewrite as

$$E_t \begin{bmatrix} s_{t+1} \\ e_{t+1} \end{bmatrix} = T \begin{bmatrix} s_t \\ e_t \end{bmatrix}$$

Where s_t $n_1 \times 1$ and e_t is $n_2 \times 1$. T is by construction upper triangular and the solution for $E_t e_{t+1}$ is hence independent of s_t. Separate the two blocks using

$$\begin{aligned} E_t s_{t+1} &= T_{ss} s_t + T_{se} e_t \\ E_t e_{t+1} &= T_{ee} e_t. \end{aligned}$$

Note that on the diagonal of T_{ee} we have all the eigenvalues with the property that $\lambda_i > 1$. If we solve the second equation forward, we notice that

$$E_t e_T = (T_{ee})^{T-t} e_t$$

and hence that $e_t = 0$ for all t is the only possible stationary equilibrium since otherwise e_t will explode (since the diagonal elements of $(T_{ee})^j$ grows towards infinity). We hence conclude that $e_t = 0$. Plugging this into the first equation, we see that

$$E_t s_{t+1} = T_{ss} s_t$$

which is a statinoary solution for s_t. Multiplying (3) with Z we recover

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = Z \begin{bmatrix} s_t \\ e_t \end{bmatrix} = \begin{bmatrix} Z_{ss} & Z_{se} \\ Z_{es} & Z_{ee} \end{bmatrix} \begin{bmatrix} s_t \\ e_t \end{bmatrix} = \begin{bmatrix} Z_{ss} \\ Z_{es} \end{bmatrix} s_t \tag{4}$$

where the last equality follows since $e_t = 0$ for all t, such that

$$\begin{aligned} x_{1,t} &= Z_{ss} s_t \quad (5) \\ x_{2,t} &= Z_{es} s_t \quad (6) \end{aligned}$$

where $Z_{ss} = Z(1:n1, 1; n1)$ and $Z_{es} = Z(n+1:n, 1:n_1)$.

Initial conditions: The initial state is assumed to be $x_{1,0}$. From the above we see that this implies

$$s_0 = Z_{ss}^{-1} x_{1,0} \quad (7)$$

.

Finally, we put the innovations back.

$$x_{1,t+1} = E_t x_{1,t+1} + \varepsilon_{t+1}$$

We use (5) to rewrite this as

$$\begin{aligned} Z_{ss} s_{t+1} &= E_t Z_{ss} s_{t+1} + \varepsilon_{t+1} \\ s_{t+1} &= T_{ss} s_t + Z_{ss}^{-1} \varepsilon_{t+1}. \quad (8) \end{aligned}$$

(7) together with (8) now represents a full solution for s. Finally, to recover x_1 we use (5). We can write

$$Z_{ss}^{-1} x_{1,t+1} = T_{ss} Z_{ss}^{-1} x_{1,t} + Z_{ss}^{-1} \varepsilon_{t+1}$$

or finally

$$\begin{aligned} x_{1,t+1} &= Z_{ss} T_{ss} Z_{ss}^{-1} + \varepsilon_{t+1} \\ x_{2,t} &= Z_{es} Z_{ss}^{-1} x_{1,t}. \end{aligned}$$

We sum this up:

Theorem 6 *Given a system of forward difference equations*

$$\begin{bmatrix} x_{1,t+1} \\ E_t x_{2,t+1} \end{bmatrix} = A \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

with initial condition $x_{1,0}$, and where x_1 contains n_1 state variables and x_2 contains n_2 jump variables. An ordered Schur decomposition of A is then two matrices Z and T such that $A = ZTZ^H$, where the sorted eigenvalues are on the diagonal of T. If and only if the number of eigenvalues with modulus smaller than unity equals n_1, the unique stable solution to the system is given by

$$\begin{aligned} x_{1,t+1} &= M + \varepsilon_{t+1} \\ x_{2,t} &= C x_{1,t}, \end{aligned}$$

where

$$\begin{aligned} Z_{ss} &= Z\left(1:n_1, 1:n_1\right) \\ Z_{es} &= Z(n_1+1:n, 1:n_1) \\ T_{ss} &= T(1:n_1, 1:n_1) \\ M &= Z_{ss} T_{ss} Z_{ss}^{-1} \\ C &= Z_{es} Z_{ss}^{-1} \end{aligned}$$

Example 7 *Solving the simple model with the Schur decomposition. The model is again*

$$\begin{bmatrix} x_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\beta^{-1}\kappa & \beta^{-1} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

such thatSince this is a diagonal matrix, we immediately notice that the eigenvalues are ρ *and* β^{-1}*. Assuming* $\kappa = 0.015$*,* $\beta = 0.99$ *and* $\rho = 0.5$ *such that we can summarize the model with a system matrix*

$$A = \begin{bmatrix} 0.5 & 0 \\ -0.99^{-1} * 0.015 & 0.99^{-1} \end{bmatrix}.$$

The Schur decomposition of A *is*

$$\begin{aligned} T &= \begin{bmatrix} 0.5 & 0.0152 \\ 0 & 1.0101 \end{bmatrix} \\ Z &= \begin{bmatrix} 0.9996 & -0.0297 \\ 0.0297 & 0.9996 \end{bmatrix} \end{aligned}$$

We first notice that the number of stable eigenvalues is 1 *which also equals the number of state variables (*x_t*). Hence, theorem* 1 *applies and the unique solution is found through*

$$\begin{aligned} Z_{ss} &= 0.9996 \\ Z_{es} &= 0.0297 \\ T_{ss} &= 0.5 \\ M &= Z_{ss}T_{ss}Z_{ss}^{-1} = 0.9996 * 0.5 * 0.9996^{-1} = 0.5 \\ C &= Z_{es}Z_{ss}^{-1} = 0.0297 * 0.9996^{-1} = 0.0297 \end{aligned}$$

Computing C *from our analytical solution above, we see that* $C = \kappa/(1 - \beta\rho) = 0.015/(1 - 0.99 * 0.5) = 0.0297$*, the identical answer. The Matlab code that computes the general model solution is surprisingly simple:*

```
[Z,T] = schur(A,'complex');
[Z,T] = ordschur(Z,T,'udi')
n = sum(abs(diag(T)));
Tss = T(1:n,1:n)
Zss = Z(1:n,1:n)
Zes = Z(n+1:end,1:n)
iZss = Zss\eye(n)
M = Zss*Tss*iZss;
C = Zes*iZss;
```