Analytics of the New-Keynesian model

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1 The model

$$\max_{C_t, H_t, B_{t+1}} E_t \sum_{T=t}^{\infty} U\left(C_T, H_T; \xi_T\right)$$

$$s.t.\ P_t C_t + B_{t+1} = W_t H_t + I_{t-1} B_t + \Pi_t + \tau_t Y_t + \int_0^1 D_t(i)\, di$$

where

$$U\left(C, H; \xi\right) = \frac{C^{1-\sigma^{-1}}}{1-\sigma^{-1}} - \frac{1}{1+v} \int_0^1 H_i^{1+v} di$$

Notice that the coefficient of relative risk-aversion is

$$-\frac{U''(C)}{U'(C)} C = -\left(\frac{-\sigma \bar{C} C^{-\sigma-1}}{\bar{C} C^{-\sigma}} C\right) = \sigma.$$

If we look at the limit when σ approaches one we get log utility, which we now impose.

The first order contions with respect to C, H, B are given by

$$\begin{aligned} \frac{1}{C_t} &= \lambda_t P_t \\ H_{t,i}^v &= \lambda_t W_t^i \\ \lambda_t &= \beta E_t \lambda_{t+1} I_t \end{aligned}$$

where λ_t is the lagrangian multiplier on the budget constraint for period t. Dividing the first two equations states that

$$\frac{W_t^i}{P_t} = H_{t,i}^v C_t \tag{1}$$

and hence that the real wage must equal the marginal rate of transformation between consumption and leisure. The third equation can be expressed as

$$\begin{aligned} 1 &= E_t\left(Q_{t,t+1} I_t\right) \\ Q_{t,t+1} &= \beta \frac{C_t}{C_{t+1}} \Pi_{t+1}^{-1} \end{aligned}$$

The firms maximize profits subject to the constraint that they can only revise their price in each period with probability $1-\alpha$.

In each period, the profit of firm i is

$$\begin{aligned} D_t^i &= p_t^i Y_t^i - TC_t^i \\ &= p_t^i Y_t^i - W_t^i H_t^i \end{aligned}$$

Theorem 1 *The Dixit Stieglitz preferences*

$$C_t = \left(\int_0^1 C_t\left(i\right)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}}$$

implies that individual demand will equal

$$Y_t^i = \left(\frac{p_t^i}{P_t}\right)^{-\theta} Y_t.$$

and that aggregate expenditures can be expressed as $P_t C_t$ *with*

$$P_t = \left(\int_0^1 P_t\left(i\right)^{1-\theta} di\right)^{\frac{1}{1-\theta}}$$

Proof. Consider the consumer minimization problem of selecting $C_t\left(i\right)$ optimally to minimize the cost of buying C units of the consumption aggregate. We omit the time subscript since this is a static problem.

$$\min_{C(i)} \int_0^1 P\left(i\right) C\left(i\right) di$$

$$s.t. \left(\int_0^1 C_t\left(i\right)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}} = C$$

The Lagrangian is

$$L = \int\limits_0^1 P(i)\, C(i)\, di - \lambda \left(\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}}\, di \right)^{\frac{\theta}{\theta-1}} - C \right).$$

The idea is now to take first order conditions with respect to two of the goods, i and j. Use one of the goods as reference, and express all first-order conditions in relation to the reference good. Next, replace all goods in the constraint, which finally gives a solution for the reference good. Finally, use that solution to solve for all the other goods.

$$\begin{aligned}
P(i) - \lambda \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}}\, di \right)^{\frac{\theta}{\theta-1}-1} (C(i))^{-\frac{1}{\theta}} &= 0 \\
P(j) - \lambda \left(\int_0^1 C(i)^{\frac{\theta-1}{\theta}}\, di \right)^{\frac{\theta}{\theta-1}-1} (C(j))^{-\frac{1}{\theta}} &= 0
\end{aligned}$$

Divide these two equations to get

$$\begin{aligned}
\frac{P(i)}{P(j)} &= \left(\frac{C(i)}{C(j)} \right)^{-\frac{1}{\theta}} \\
C(i) &= \left(\frac{P(i)}{P(j)} \right)^{-\theta} C(j)
\end{aligned}$$

Insert this into the constraint to get

$$\left(\int_0^1 \left(\left(\frac{P(i)}{P(j)} \right)^{-\theta} C(j) \right)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} = C$$

Since $P(j)$ and $C(j)$ are the same in the integral they can be moved outside to get

$$\begin{aligned}
C &= C(j)\, (P(j))^{\theta} \left(\int_0^1 \left((P(i))^{1-\theta} \right) di \right)^{\frac{\theta}{\theta-1}} \\
C(j) &= \frac{P(j)^{-\theta}}{\left(\int_0^1 \left((P(i))^{1-\theta} \right) di \right)^{\frac{\theta}{\theta-1}}} C. \qquad (2)
\end{aligned}$$

Hence, the minimum expenditure required to by C units of the consumption good is equal to

$$\begin{aligned}\int_0^1 P(j)\,C(j)\,dj &= \int_0^1 P(j)\frac{P(j)^{-\theta}}{\left(\int_0^1\left((P(i))^{1-\theta}\right)di\right)^{\frac{\theta}{\theta-1}}}Cdj\\ &= \frac{\int_0^1 P(j)^{1-\theta}\,dj}{\left(\int_0^1\left((P(i))^{1-\theta}\right)di\right)^{\frac{\theta}{\theta-1}}}C\end{aligned}$$

Next, notice that the integral in the nominator and denominator is identical, such that

$$\begin{aligned}\int_0^1 P(j)\,C(j)\,dj &= \left(\int_0^1 P(i)^{1-\theta}\,di\right)^{1-\frac{\theta}{\theta-1}} C\\ &= \left(\int_0^1 P(i)^{1-\theta}\,di\right)^{\frac{1}{1-\theta}} C.\end{aligned}$$

We now define the price index P as the minimum expenditure to buy one unit of the consumption good, such that

$$P = \left(\int_0^1 P(i)^{1-\theta}\,di\right)^{\frac{1}{1-\theta}}.$$

We now see that if we substitute this definition into (2), we get

$$C(j) = \left(\frac{P(j)}{P}\right)^{-\theta} C.$$

■

The production function is $Y_t^i = A_t H_t^i$ such that the amount of hours needed to produce a given volume of output is

$$H_t^i = \frac{Y_t^i}{A_t}. \tag{3}$$

Substituting these expressions into the profit function gives

$$D_t^i = p_t^i \left(\frac{p_t^i}{P_t}\right)^{-\theta} Y_t - W_t^j \frac{1}{A_t} \left(\frac{p_t^i}{P_t}\right)^{-\theta} Y_t$$

Note the j on the wage in the previous expression. We assume that each good is produced by a large number of firms. This is to indicate that firm i is so small that it's decision has no impact on the wage for the type of labour needed to produce its output. In contrast, if it was the only firm in the sector, it would internalize that to produce an aditional unit out output, the wage would have to be raised a bit in order to attract enough labour of the required type. Next, we discount the future with the stochastic discount factor as well as with the Calvo-probability and look at the expected profits during the expected life of the price.

$$\begin{aligned}
\frac{\partial}{\partial p_t^i} E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(p_t^i \left(\frac{p_t^i}{P_T}\right)^{-\theta} Y_T - W_T^j \frac{1}{A_T} \left(\frac{p_t^i}{P_T}\right)^{-\theta} Y_T \right) &= 0 \\
\frac{\partial}{\partial p_t^i} E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(\left(p_t^i\right)^{1-\theta} P_T^{\theta} Y_T - \left(p_t^i\right)^{-\theta} W_T^j \frac{1}{A_T} \left(\frac{1}{P_T}\right)^{-\theta} Y_T \right) &= 0 \\
E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left((1-\theta) \left(\frac{p_t^i}{P_T}\right)^{-\theta} Y_T + \theta \left(p_t^i\right)^{-\theta-1} W_T^j \frac{1}{A_T} \left(\frac{1}{P_T}\right)^{-\theta} Y_T \right) &= 0 \\
E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(\left(p_t^i\right)^{-\theta-(-\theta-1)} (1-\theta) \left(\frac{1}{P_T}\right)^{-\theta} Y_T + \theta W_T^j \frac{1}{A_T} \left(\frac{1}{P_T}\right)^{-\theta} Y_T \right) &= 0 \\
E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(\left(p_t^i\right) (1-\theta) \left(\frac{1}{P_T}\right)^{-\theta} Y_T + \theta W_T^j \frac{1}{A_T} \left(\frac{1}{P_T}\right)^{-\theta} Y_T \right) &= 0 \qquad (4)
\end{aligned}$$

Next, we substitute (3) into (1) to get

$$W_t^j = P_t \left(\frac{Y_t^j}{A_t}\right)^{v} C_t$$

and use that all firms in sector j sets the same price, equal to p_t^i, and hence get demand

$$\begin{aligned}
W_t^j &= P_t \left(\frac{1}{A_t} \left(\frac{p_t^i}{P_t}\right)^{-\theta} Y_t \right)^{v} C_t \\
W_t^j &= P_t \left(p_t^i\right)^{-\theta v} \left(\frac{1}{A_t} P_t^{\theta} Y_t \right)^{v} C_t \qquad (5)
\end{aligned}$$

(5) into (4) gives

$$
\begin{aligned}
E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(p_t^i (1-\theta) P_T^{\theta} Y_T + \theta P_T \left(p_t^i\right)^{-\theta v} \left(\frac{1}{A_T} P_T^{\theta} Y_T \right)^{v} C_T \frac{1}{A_T} P_T^{\theta} Y_T \right) &= 0 \\
E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(p_t^i (1-\theta) P_T^{\theta} Y_T + \theta P_T \left(p_t^i\right)^{-\theta v} C_T \left(\frac{1}{A_T} P_T^{\theta} Y_T \right)^{1+v} \right) &= 0 \\
E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(\left(p_t^i\right)^{1+\theta v} (1-\theta) P_T^{\theta} Y_T + \theta P_T C_T \left(\frac{1}{A_T} P_T^{\theta} Y_T \right)^{1+v} \right) &= 0
\end{aligned}
$$

$$
\left(p_t^i\right)^{1+v\theta} E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} P_T^{\theta} Y_T = \frac{\theta}{\theta-1} E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(P_T^{1+\theta(1+v)} C_T \left(\frac{Y_T}{A_T} \right)^{1+v} \right)
$$

Next, substitute the stochastic discount factor

$$
\left(p_t^i\right)^{1+v\theta} E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{C_t P_t}{C_T P_T} P_T^{\theta} Y_T = \frac{\theta}{\theta-1} E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{C_t P_t}{C_T P_T} \left(P_T^{1+\theta(1+v)} C_T \left(\frac{Y_T}{A_T} \right)^{1+v} \right)
$$

The term $C_t P_t$ appears in all terms in both sums and hence drops out. We can hence write this as

$$
\left(p_t^i\right)^{1+v\theta} E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \frac{1}{C_T} P_T^{\theta-1} Y_T = \frac{\theta}{\theta-1} E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left(P_T^{\theta(1+v)} \left(\frac{Y_T}{A_T} \right)^{1+v} \right) \tag{6}
$$

Finally, we note that $1 + v\theta + \theta - 1 = \theta(1+v)$, and thus if we divide both sides with $P_t^{\theta(1+v)}$ and use that $C_t = Y_t$ we get

$$
\begin{aligned}
\left(\frac{p_t^i}{P_t} \right)^{1+v\theta} &= \frac{K_t}{F_t} \qquad (7) \\
K_t &= E_t \frac{\theta}{\theta-1} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left(\left(\frac{P_T}{P_t} \right)^{\theta(1+v)} \left(\frac{Y_T}{A_T} \right)^{1+v} \right) \\
F_t &= E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left(\frac{P_T}{P_t} \right)^{\theta-1}.
\end{aligned}
$$

These expressions are found in Woodford (2011) eqn (50) - (54), apart from the fact that I have already used log-preferences from the start and have hence assumed $\tilde{\sigma} = 1$ in his notation.

Let us summarize: (7) corresponds to a non-linear Phillips curve where the optimal price set today is related to current and expected future marginal cost conditions.

These equations can be recursified. Starting with F, note that

$$F_{t+1} = E_{t+1} \sum_{T=t+1}^{\infty} (\alpha\beta)^{T-(t+1)} (1-\tau_T) \left(\frac{P_T}{P_{t+1}}\right)^{\theta-1}.$$

If we multiply this with $\alpha\beta\left(\frac{P_{t+1}}{P_t}\right)^{\theta-1}$ and take expectation at t

$$E_t \alpha\beta \left(\frac{P_{t+1}}{P_t}\right)^{\theta-1} F_{t+1} = E_t E_{t+1} \sum_{T=t+1}^{\infty} (\alpha\beta)^{T-t} \left(\frac{P_T}{P_t}\right)^{\theta-1}.$$

But due to the law of itterated expectations, the right-hand side here is equivalent to F_t, apart from the term when $T = t$. Hence,

$$F_t = 1 + \alpha\beta E_t \left(\Pi_{t+1}^{\theta-1} F_{t+1}\right) \tag{8}$$

where we have defined $\Pi_t = P_t/P_{t-1}M$.

Similarly, by multiplying K_{t+1} with $\alpha\beta\Pi_{t+1}^{\theta(1+\omega)}$ and taking expectations and subtracting K_t we get that

$$K_t = \frac{\theta}{\theta-1}\left(\frac{Y_t}{A_t}\right)^{1+v} + \alpha\beta E_t \Pi_{t+1}^{\theta(1+v)} K_{t+1} \tag{9}$$

Next, we use the definition of the price-index to link changes in the optimal price to changes in the inflation rate. The logic is that only the firms that actually change prices contibutes to inflation, since the other prices are fixed. Since the chance of changing the price is equal for all firms, the integral of the unchanged part of the price-index must equal the previous periods price-index, adjusted for the "thinning out". Hence, from the definition of the price index,

$$P_t^{1-\theta} = \int_0^1 \left(P_t^i\right)^{1-\theta} di = (1-\alpha)\left(p_t^*\right)^{1-\theta} + \alpha P_{t-1}^{1-\theta}$$

Dividing through with $P_t^{1-\theta}$ gives

$$1 = (1-\alpha)\left(\frac{p_t^*}{P_t}\right)^{1-\theta} + \alpha\Pi_t^{\theta-1} \tag{10}$$

Substituting (7) into (10) gives

$$1 = (1-\alpha)\left(\frac{K_t}{F_t}\right)^{\frac{1-\theta}{1+\omega\theta}} + \alpha\Pi_t^{\theta-1}$$

To summarize, the full set of first order conditions can be summarized by

$$\begin{aligned}
1 &= (1-\alpha)\left(\frac{K_t}{F_t}\right)^{\frac{1-\theta}{1+\omega\theta}} + \alpha\Pi_t^{\theta-1} \qquad (11)\\
F_t &= 1 + \alpha\beta E_t\left(\Pi_{t+1}^{\theta-1}F_{t+1}\right)\\
K_t &= \frac{\theta}{\theta-1}\left(\frac{Y_t}{A_t}\right)^{1+v} + \alpha\beta E_t\Pi_{t+1}^{\theta(1+v)}K_{t+1}\\
\frac{1}{I_t} &= \beta E_t\left(\frac{Y_t}{Y_{t+1}}\Pi_{t+1}^{-1}\right)
\end{aligned}$$

This amounts to 4 equations for 5 endogenous variables. Something needs to pin down the nominal side of the economy, and we will close the model with a Taylor-type rule.

$$I_t = I\left(\frac{\Pi_t}{\Pi^*}\right)^{\chi}\exp^{e_t}.$$

1.1 The steady state

We start by examining the steady state of the above system. The fourth eqation gives

$$I = \beta^{-1}\Pi$$

The policy rule then gives

$$\Pi = \Pi^*$$

since by assumption the steady state level of the shock is zero. Lets work with $\Pi^* = 1$. Then the first equation implies

$$1 = \left(\frac{K}{F}\right)^{\frac{1-\theta}{1+\omega\theta}}$$

$$\begin{aligned}
F &= \frac{1}{1-\alpha\beta}\\
K &= \frac{1}{1-\alpha\beta}\frac{\theta}{\theta-1}Y^{1+v}
\end{aligned}$$

Dividing K with F and inserting this into the first equation gives

$$\begin{aligned}\frac{1}{1-\alpha\beta} &= \frac{1}{1-\alpha\beta}\frac{\theta}{\theta-1}Y^{1+v} \\ 1 &= \frac{\theta}{\theta-1}Y^{1+v}\end{aligned}$$

This pinns down Y as a function of the model parameters.

$$Y = \left(\frac{(\theta-1)}{\theta}\right)^{\frac{1}{1+v}}$$

1.2 The flex-price equilibrium

Let us assume that prices are fully flexible. Going back to equations (7), with $\alpha = 0$ this reduces to

$$\begin{aligned}\left(\frac{p_t^i}{P_t}\right)^{1+v\theta} &= \frac{K_t}{F_t} \\ K_t &= \frac{\theta}{\theta-1}\left(\left(\frac{Y_t}{A_t}\right)^{1+v}\right) \\ F_t &= 1.\end{aligned}$$

Since prices are flexible, $p_t^i = P_t$ and we get

$$\frac{\theta}{\theta-1}\left(\left(\frac{Y_t^n}{A_t}\right)^{1+v}\right) = 1 \tag{12}$$

This equation implicity defines Y_t^n as a function of the exogenous shocks.

Next, consider a one-period asset that at time $t+1$ delivers $\frac{P_{t+1}}{P_t}$ dollars, which hence fully insures its owner against inflation. The return on this asset is given by

$$\begin{aligned}\frac{1}{R_t^n} &= E_t\left(Q_{t,t+1}\frac{P_{t+1}}{P_t}\right) \\ \frac{1}{R_t^n} &= E_t\left(\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{P_{t+1}}{P_t}\right) \\ \frac{1}{R_t^n} &= E_t\left(\beta\frac{Y_t}{Y_{t+1}}\right) \qquad (13)\end{aligned}$$

Here, R_t^n is the natural real rate. The equations (12) and (13) together define the flex-price equilibrium in this economy. Given these definition, it is now possible to examine a different policy rule, of the kind

$$I_t = R_t^n \left(\frac{\Pi_t}{\Pi^*}\right)^{\chi} \exp \varepsilon_t^i.$$

In an equilibrium with zero steady-state inflation and an efficient production subsidy that eliminates the inefficiently low output associated with monopolistic competition, this will amount to stabilizing output at its natural rate and lead to zero inflation, for all shocks except the monetary policy shock.

1.2.1 Log-linearization

1.3 Preliminaries

I think of log-linearizations the following way. We want to end up with a model approximated in log-deviations from steady state. That is, the non-linear equation $Y_t = f(X_t)$ should be transformed into $\hat{Y}_t = c\hat{X}_t$, where $\hat{Y}_t = \log Y_t - \log Y$, where Y is the steady state value of Y_t. I think the most precise way to introduce log-linearization is to rewrite the equation the following way:

$$\exp(\log Y_t) = f(\exp(\log X_t))$$

which is fine for all variables that are strictly greater than zero. Then, define the new variable $y_t = \log Y_t$ etc. and write

$$\exp(y_t) = f(\exp x_t)$$

with $y = \log Y$ and $x = \log X$. Now linearize this equation around y and x. A first-order Taylor expansion gives

$$\begin{aligned} \exp(y)(y_t - y) &= \exp(x) f'(\exp x)(x_t - x) \\ Y(y_t - y) &= X f'(X)(x_t - x) \end{aligned}$$

Depending on the shape of f, sometimes the steady state relationship can be used to simplify the above expression.

We can alternatively skip the rewriting step and instead directly take derivatives with respect to $\log Y_t$, using the rules

$$\begin{aligned} \frac{\partial Y_t}{\partial \log Y_t} &= \frac{\partial \exp\left(\log Y_t\right)}{\partial \log Y_t} = Y_t \\ \frac{\partial f\left(Y_t\right)}{\partial \log Y_t} &= f'\left(Y_t\right) \frac{\partial Y_t}{\partial \log Y_t} = f'\left(Y_t\right) Y_t \end{aligned} \tag{14}$$

1.3.1 Log-linearizing the model equations

$$\begin{aligned} 1 &= (1-\alpha)\left(\frac{K_t}{F_t}\right)^{\frac{1-\theta}{1+v\theta}} + \alpha \Pi_t^{\theta-1} \\ F_t &= 1 + \alpha\beta E_t\left(\Pi_{t+1}^{\theta-1} F_{t+1}\right) \\ K_t &= \frac{\theta}{\theta-1}\left(\frac{Y_t}{A_t}\right)^{1+v} + \alpha\beta E_t \Pi_{t+1}^{\theta(1+v)} K_{t+1} \\ \frac{1}{I_t} &= \beta E_t\left(\frac{Y_t}{Y_{t+1}} \Pi_{t+1}^{-1}\right) \end{aligned} \tag{15}$$

We start by log-linearizing the equation for F.

We need to think carefully about the way we treat the shocks. With the notation above, A_t is the level of productivity. We can now proceed to the calculations. First F:

$$\begin{aligned} F\hat{F}_t &= \alpha\beta E_t\left((\theta-1) F\hat{\Pi}_{t+1} + F\hat{F}_{t+1}\right) \\ \hat{F}_t &= \alpha\beta E_t\left((\theta-1)\hat{\Pi}_{t+1} + \hat{F}_{t+1}\right) \end{aligned} \tag{16}$$

$$\hat{K}_t = (1-\alpha\beta)(1+v)\left(\hat{Y}_t - \hat{A}_t\right) + \alpha\beta E_t\left(\theta(1+\omega)\hat{\Pi}_{t+1} + \hat{K}_{t+1}\right). \tag{17}$$

The $1 = (1-\alpha)\left(\frac{K_t}{F_t}\right)^{\frac{1-\theta}{1+v\theta}} + \alpha\Pi_t^{\theta-1}$ equation can be written

$$\left(\frac{1-\alpha\Pi_t^{\theta-1}}{(1-\alpha)}\right)^{\frac{1+v\theta}{1-\theta}} = \frac{K_t}{F_t}$$

from which we can see that in a zero-inflation steady state, $K = F$. Log-linearization gives

$$
\begin{aligned}
-\frac{\alpha(\theta-1)}{1-\alpha}\frac{1+v\theta}{1-\theta}\left(\frac{1-\alpha\Pi^{\theta-1}}{(1-\alpha)}\right)^{\frac{1+v\theta}{1-\theta}-1}\hat{\Pi}_t &= \frac{1}{F}K\hat{K}_t - \frac{K}{F^2}F\hat{F}_t \\
\frac{\alpha(1+v\theta)}{1-\alpha}\hat{\Pi}_t &= \hat{K}_t - \hat{F}_t \qquad (18)
\end{aligned}
$$

Subtracting (16) from (17) yields

$$
\hat{K}_t - \hat{F}_t = (1-\alpha\beta)(1+v)\left(\hat{Y}_t - \hat{A}_t\right) + \alpha\beta E_t\left((1+\theta v)\hat{\Pi}_{t+1} + \hat{K}_{t+1} - \hat{F}_{t+1}\right)
$$

Using (18) in the above gives

$$
\begin{aligned}
\frac{\alpha(1+v\theta)}{1-\alpha}\hat{\Pi}_t &= (1-\alpha\beta)(1+v)\left(\hat{Y}_t - \hat{A}_t\right) + \alpha\beta E_t\left((1+\theta\omega)\hat{\Pi}_{t+1} + \frac{\alpha(1+\omega\theta)}{1-\alpha}\hat{\Pi}_{t+1}\right) \\
\hat{\Pi}_t &= (1-\alpha)(1-\alpha\beta)\frac{(1+v)}{\alpha(1+v\theta)}\left(\hat{Y}_t - \hat{A}_t\right) + \frac{(1-\alpha)\alpha\beta}{\alpha(1+v\theta)}E_t\left((1+\theta v)\hat{\Pi}_{t+1} + \frac{\alpha(1+v\theta)}{1-\alpha}\hat{\Pi}_{t+1}\right) \\
\hat{\Pi}_t &= \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{(1+v)}{(1+v\theta)}\left(\hat{Y}_t - \hat{A}_t\right) + \beta E_t\left((1-\alpha)\hat{\Pi}_{t+1} + \alpha\hat{\Pi}_{t+1}\right)
\end{aligned}
$$

wich finally allows us to write

$$
\hat{\Pi}_t = \kappa\left(\hat{Y}_t - \hat{A}_t\right) + \beta E_t\hat{\Pi}_{t+1}.
$$

It is instructive to redo these calculations for the flex-price case.

1.3.2 Log-linearizing the flex-price case

With $a = 0$, we write the equations using Woodfords notation above as

$$
\begin{aligned}
\left(\frac{p_t^i}{P_t}\right)^{1+v\theta} &= \frac{K_t}{F_t} \\
K_t &= \frac{\theta}{\theta-1}\left(\frac{Y_t^n}{A_t}\right)^{1+v} \\
F_t &= 1
\end{aligned}
$$

Since the relative price in the flex-price equilibrium will equal one by definition, we get that Y_t^n is the solution to

$$
1 = \frac{\theta}{\theta-1}\left(\frac{Y_t^n}{A_t}\right)^{1+v}
$$

Taking logs and subtracting steady state,

$$\hat{Y}_t^n = \hat{A}_t$$

which inserted in (**??**) finally gives

$$\hat{\Pi}_t = \kappa \left(\hat{Y}_t - \hat{Y}_t^n \right) + \beta E_t \hat{\Pi}_{t+1}.$$